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J. Phys. A: Math. Gen. 19 (1986) 191-203. Printed in Great Britain

# On the exactly soluble model in quantum electrodynamics

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Received 16 January 1985, in final form 6 June 1985

**Abstract.** The model of a ladder configuration three-level atom interacting with a two-mode near-resonant radiation field is treated. It is shown that the operator equations of motion can be solved explicitly. The dynamical behaviour of the photon numbers and level populations is studied for various initial conditions.

## 1. Introduction

The problem of the three-level atom interacting with the electromagnetic field is the object of much research activity over the last ten years. It is central to discussions of two-photon coherence [1, 2], resonance Raman scattering and double-resonance processes [3], three-level super-radiance [4, 5], two-mode laser [6], three-level echoes [7] and population dynamics and spectra of a driven three-level system [8-11].

A number of recent papers have been dedicated to a careful consideration of the problem of dynamics of a single three-level atom interacting with two resonant modes of the radiation field. The semiclassical formalism for the treatment of this problem has been discussed [8, 9, 12, 13]. In another series of articles [6, 11, 14, 15] the fully quantised theory has been studied. Exact Schrödinger wavefunctions have been obtained for some special initial states [6, 11]. In the work of Li and Bei [14] the explicit expression of the evolution operator has been derived in the interaction picture for the case of exact one-photon resonance. The rigorous examination of the dynamical behaviour of level populations and photon numbers has been realised in the Heisenberg picture by Bogolubov *et al* [15] for the three-level two-photon lambda configuration. On the other hand, the exact solution of the nonlinear equation for the energy operator of a few-level atom interacting with a single-mode radiation field has been obtained by Buck and Sukumar [16]. In this paper we shall show that the operator equations for the three-level two-photon ladder configuration detuned from one-photon resonance can be solved explicitly. By using the exact solution obtained here we shall examine the dynamical behaviour of photon numbers and level populations for arbitrary initial states of the field.

The remainder of this paper is organised as follows. Section 2 gives the model Hamiltonian and § 3 shows the exact solution of operator equations for level populations and photon numbers. In § 4 the time evolution of photon numbers and level populations in the case of quantum initial states is considered, while § 5 gives the time evolution of photon numbers and level populations in the case of an arbitrary initial field. We give a summary in § 6.

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## 2. The model Hamiltonian

We consider a three-level atom of ladder configuration (see figure 1) in which non-zero dipole moments exist only between levels 1 and 3, and between 2 and 3. The dipole transition between levels 1 and 2 is thus forbidden. Let the atom be at rest in a lossless cavity and interact with a two-mode radiation field. The energy operator for the atom is

$$\hat{H}_{\mathrm{A}} = \sum_{j=1}^{3} \hbar\Omega_j \hat{R}_{jj}. \tag{1}$$

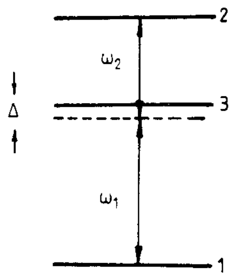


**Figure 1.** A ladder configuration three-level atom interacting with a two-mode near-resonant radiation field.

Here, the operator $\hat{R}_{jj} = |j\rangle\langle j|$ describes the population of level $j$ and $\hbar\Omega_j$ is the corresponding energy. The atomic eigenstate vectors $|j\rangle$ $(j = 1, 2, 3)$ form the basis of the state space of the three-level atom

$$\begin{gathered} \hat{H}_{\mathrm{A}}|j\rangle = \hbar\Omega_j|j\rangle \\ \langle i|j\rangle = \delta_{ij} \qquad \sum_{j=1}^{3} |j\rangle\langle j| = 1. \end{gathered} \tag{2}$$

The field Hamiltonian is

$$\hat{H}_{\mathrm{F}} = \sum_{\alpha=1}^{2} \hbar\omega_\alpha \hat{a}_\alpha^{+} \hat{a}_\alpha. \tag{3}$$

The operators $\hat{a}_\alpha$, $\hat{a}_\alpha^{+}$ describe near-resonant mode $\alpha$ of the radiation field in the cavity. The corresponding frequencies of the modes are $\omega_\alpha$, where $|(\omega_\alpha - |\Omega_3 - \Omega_\alpha|)| \ll \omega_\alpha$. The atom–field interaction is described in the dipole and rotating wave approximations by [17]

$$\hat{H}_{\mathrm{AF}} = \hbar g_1(\hat{a}_1 \hat{R}_{31} + \hat{a}_1^{+} \hat{R}_{13}) + \hbar g_2(\hat{a}_2 \hat{R}_{23} + \hat{a}_2^{+} \hat{R}_{32}). \tag{4}$$

Here, the operator $\hat{R}_{ij} = |i\rangle\langle j|$ describes the atomic transition from level $j$ to level $i$ $(i \neq j)$. The parameters $g_\alpha$ are the constants of atom-mode coupling. Thus, the total model Hamiltonian of the 'atom–field' system is

$$\begin{aligned} \hat{H} &= \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{F}} + \hat{H}_{\mathrm{AF}} \\ &= \sum_{j=1}^{3} \hbar\Omega_j \hat{R}_{jj} + \sum_{\alpha=1}^{2} \hbar\omega_\alpha \hat{a}_\alpha^{+} \hat{a}_\alpha + \hbar g_1(\hat{a}_1 \hat{R}_{31} + \hat{a}_1^{+} \hat{R}_{13}) \\ &\quad + \hbar g_2(\hat{a}_2 \hat{R}_{23} + \hat{a}_2^{+} \hat{R}_{32}). \end{aligned} \tag{5}$$

Note that the operators $\hat{R}_{ij} = |i\rangle\langle j|$ $(i, j = 1, 2, 3)$ are the generators of the group SU(3) and obey the following relations:

$$\hat{R}_{ij}\hat{R}_{kl} = \hat{R}_{il}\delta_{kj} \tag{6}$$

$$\sum_{i=1}^{3} \hat{R}_{ii} = 1. \tag{7}$$

By using (6) the commutation rules

$$[\hat{R}_{ij}, \hat{R}_{kl}] = \hat{R}_{il}\delta_{kj} - \hat{R}_{kj}\delta_{il} \tag{8}$$

are quickly established [4]. The commutation relations of the photon operators $\hat{a}_\alpha$, $\hat{a}_\alpha^+$ $(\alpha = 1, 2)$ are

$$[\hat{a}_\alpha, \hat{a}_{\alpha'}^+] = \delta_{\alpha\alpha'} \qquad [\hat{a}_\alpha, \hat{a}_{\alpha'}] = 0 \qquad [\hat{a}_\alpha^+, \hat{a}_{\alpha'}^+] = 0. \tag{9}$$

Assuming that there is exact two-photon resonance, the detuning parameter $\Delta$ can be defined as

$$\Delta = (\Omega_3 - \Omega_1) - \omega_1 = \omega_2 - (\Omega_2 - \Omega_3). \tag{10}$$

## 3. Exact solution of operator equations for level populations and photon numbers

Starting from the Hamiltonian (5) and the commutators (8) and (9) we write down the Heisenberg equations for various operators in the usual way, i.e., $\dot{\hat{O}} = (\mathrm{i}/\hbar)[\hat{H}, \hat{O}]$. It is convenient to define the subsidiary operators

$$\begin{aligned} \hat{A}_1 &\equiv \mathrm{i}(\hat{a}_1\hat{R}_{31} - \hat{a}_1^+\hat{R}_{13}) \\ \hat{A}_2 &\equiv \mathrm{i}(\hat{a}_2\hat{R}_{23} - \hat{a}_2^+\hat{R}_{32}). \end{aligned} \tag{11}$$

Then the Heisenberg equations for the level population operators $\hat{R}_{\alpha\alpha}$ and the photon number operators $\hat{N}_\alpha = \hat{a}_\alpha^+\hat{a}_\alpha$ $(\alpha = 1, 2)$ are quickly established:

$$\begin{aligned} \dot{\hat{R}}_{11}(t) &= \dot{\hat{N}}_1(t) = g_1\hat{A}_1(t) \\ \dot{\hat{R}}_{22}(t) &= -\dot{\hat{N}}_2(t) = -g_2\hat{A}_2(t). \end{aligned} \tag{12}$$

It follows that

$$\begin{aligned} \hat{N}_1(t) - \hat{R}_{11}(t) &= \text{constant} \equiv \hat{M}_1 \\ \hat{N}_2(t) + \hat{R}_{22}(t) &= \text{constant} \equiv \hat{M}_2 + 1 \end{aligned} \tag{13}$$

where $\hat{M}_\alpha$ are time-independent operators.

By using the relations (6) the Heisenberg equations for $\hat{A}_\alpha$ are found to be

$$\begin{aligned} \dot{\hat{A}}_1(t) &= -\Delta\hat{C}_1(t) + 2g_1(\hat{M}_1 + 1)[1 - 2\hat{R}_{11}(t) - \hat{R}_{22}(t)] - g_2\hat{B}(t) \\ \dot{\hat{A}}_2(t) &= \Delta\hat{C}_2(t) - 2g_2(\hat{M}_2 + 1)[1 - 2\hat{R}_{22}(t) - \hat{R}_{11}(t)] + g_1\hat{B}(t) \end{aligned} \tag{14}$$

where

$$\begin{aligned} \hat{B} &\equiv \hat{a}_1\hat{a}_2\hat{R}_{21} + \hat{a}_1^+\hat{a}_2^+\hat{R}_{12} \\ \hat{C}_1 &\equiv \hat{a}_1\hat{R}_{31} + \hat{a}_1^+\hat{R}_{13} \\ \hat{C}_2 &\equiv \hat{a}_2\hat{R}_{23} + \hat{a}_2^+\hat{R}_{32}. \end{aligned} \tag{15}$$

The operators $\hat{B}$ and $\hat{C}_\alpha$ obey the following equations of motion:

$$\begin{aligned}\dot{\hat{B}}(t) &= -g_1(\hat{M}_1+1)\hat{A}_2(t)+g_2(\hat{M}_2+1)\hat{A}_1(t)\\ \dot{\hat{C}}_1(t) &= \Delta\hat{A}_1(t)+g_2\hat{D}(t)\\ \dot{\hat{C}}_2(t) &= -\Delta\hat{A}_2(t)-g_1\hat{D}(t)\end{aligned}\tag{16}$$

where

$$\hat{D}\equiv \mathrm{i}(\hat{a}_1\hat{a}_2\hat{R}_{21}-\hat{a}_1^+\hat{a}_2^+\hat{R}_{12}).\tag{17}$$

Finally, the Heisenberg equation for the operator $\hat{D}$ is found to be

$$\dot{\hat{D}}(t)=g_1(\hat{M}_1+1)\hat{C}_2(t)-g_2(\hat{M}_2+1)\hat{C}_1(t).\tag{18}$$

The equations (12), (14), (16) and (18) form a closed linear system which has the following two integrals of motion:

$$\begin{aligned}-g_1g_2\hat{B}(t)+g_1^2(\hat{M}_1+1)\hat{R}_{22}(t)+g_2^2(\hat{M}_2+1)\hat{R}_{11}(t) &= \text{constant}\equiv\hat{K}\\ g_1\hat{C}_1(t)+g_2\hat{C}_2(t)-\Delta[\hat{R}_{11}(t)+\hat{R}_{22}(t)] &= \text{constant}\equiv\hat{Q}.\end{aligned}\tag{19}$$

Here $\hat{K}$ and $\hat{Q}$ are time-independent operators. It is easy to establish that the operators $\hat{M}_\alpha$, $\hat{K}$ and $\hat{Q}$ commute with $\hat{H}$ and each other. Taking into account (19) one can obtain from (12), (14), (16) and (18) the operator equations

$$\begin{aligned}\ddot{\hat{R}}_{11}(t) &= -(3\hat{\lambda}_1^2+\hat{\lambda}_0^2)\hat{R}_{11}(t)-3\hat{\lambda}_1^2\hat{R}_{22}(t)-\Delta g_1\hat{C}_1(t)+2\hat{\lambda}_1^2+\hat{K}\\ \ddot{\hat{R}}_{22}(t) &= -(3\hat{\lambda}_2^2+\Delta^2)\hat{R}_{11}(t)-(3\hat{\lambda}_2^2+\hat{\lambda}_0^2+\Delta^2)\hat{R}_{22}(t)+\Delta g_1\hat{C}_1(t)+2\hat{\lambda}_2^2+\hat{K}-\Delta\hat{Q}\\ g_1\ddot{\hat{C}}_1(t) &= \Delta\ddot{\hat{R}}_{11}(t)+\Delta\hat{\lambda}_1^2[\hat{R}_{11}(t)+\hat{R}_{22}(t)]-\hat{\lambda}_0^2g_1\hat{C}_1(t)+\hat{\lambda}_1^2\hat{Q}.\end{aligned}\tag{20}$$

Here we have introduced the notation

$$\begin{aligned}\hat{\lambda}_\alpha &\equiv g_\alpha(\hat{M}_\alpha+1)^{1/2}\\ \hat{\lambda}_0 &\equiv (\hat{\lambda}_1^2+\hat{\lambda}_2^2)^{1/2}.\end{aligned}\tag{21}$$

To solve the system of second-order differential equations (20) we ought to determine the eigenvalues of the characteristic matrix. This leads to the following equation:

$$\det\begin{pmatrix}\hat{X}^2-(3\hat{\lambda}_1^2+\hat{\lambda}_0^2) & -3\hat{\lambda}_1^2 & -\mathrm{i}\Delta\\ -(3\hat{\lambda}_2^2+\Delta^2) & \hat{X}^2-(3\hat{\lambda}_2^2+\hat{\lambda}_0^2+\Delta^2) & \mathrm{i}\Delta\\ \mathrm{i}\Delta(\hat{\lambda}_1^2-\hat{X}^2) & \mathrm{i}\Delta\hat{\lambda}_1^2 & \hat{X}^2-\hat{\lambda}_0^2\end{pmatrix}$$
$$=\hat{X}^6-2(3\hat{\lambda}_0^2+\Delta^2)\hat{X}^4+(3\hat{\lambda}_0^2+\Delta^2)^2\hat{X}^2-\hat{\lambda}_0^4(4\hat{\lambda}_0^2+\Delta^2)=0.\tag{22}$$

The solutions of this equation are found to be $\hat{\lambda}_+$, $\hat{\lambda}_-$ and $2\hat{\lambda}$, where

$$\hat{\lambda}\equiv(\hat{\lambda}_0^2+\tfrac{1}{4}\Delta^2)^{1/2}\qquad \hat{\lambda}_+=\hat{\lambda}+\tfrac{1}{2}\Delta\qquad \hat{\lambda}_-=\hat{\lambda}-\tfrac{1}{2}\Delta.\tag{23}$$

They are the operators of the frequencies of nonlinear optical oscillations in the three-level system [9, 11, 18]. Now the solution of the system (20) can be presented in the form

$$\begin{aligned}\hat{R}_{11}(t) &= \hat{P}_+(t)+\hat{P}_-(t)+\hat{\lambda}_1^2\hat{P}(t)+\hat{R}_{11}(0)\\ \hat{R}_{22}(t) &= -\hat{P}_+(t)-\hat{P}_-(t)+\hat{\lambda}_2^2\hat{P}(t)+\hat{R}_{22}(0)\end{aligned}\tag{24}$$

where

$$\hat{P}_+(t) = \hat{\mu}_+(\cos \hat{\lambda}_+ t - 1) + \hat{\beta}_+ \sin \hat{\lambda}_+ t$$
$$\hat{P}_-(t) = \hat{\mu}_-(\cos \hat{\lambda}_- t - 1) + \hat{\beta}_- \sin \hat{\lambda}_- t \qquad (25)$$
$$\hat{P}(t) = \hat{\mu}(\cos 2\hat{\lambda} t - 1) + \hat{\beta} \sin 2\hat{\lambda} t.$$

The amplitude operators $\hat{\mu}_\pm$, $\hat{\mu}$, $\hat{\beta}_\pm$ and $\hat{\beta}$ are found to be

$$\hat{\mu} = \{\hat{\lambda}_1^2 \hat{R}_{11}(0) + \hat{\lambda}_2^2 \hat{R}_{22}(0) - \hat{\lambda}_0^2 \hat{R}_{33}(0) + g_1 g_2 \hat{B}(0) + \tfrac{1}{2}\Delta[g_1 \hat{C}_1(0) + g_2 \hat{C}_2(0)]\}/(2\hat{\lambda}_0^2 \hat{\lambda}^2)$$
$$\hat{\mu}_\pm = [\hat{R}_{11}(0) - \hat{R}_{22}(0)]\hat{\lambda}_1^2 \hat{\lambda}_2^2/(\hat{\lambda}_0^2 \hat{\lambda}\hat{\lambda}_\pm) + g_1 g_2 \hat{B}(0)(\hat{\lambda}_2^2 - \hat{\lambda}_1^2)/(2\hat{\lambda}_0^2 \hat{\lambda}\hat{\lambda}_\pm)$$
$$\pm [\hat{\lambda}_2^2 g_1 \hat{C}_1(0) - \hat{\lambda}_1^2 g_2 \hat{C}_2(0)]/(2\hat{\lambda}_0^2 \hat{\lambda}) \qquad (26)$$
$$\hat{\beta} = [g_1 \hat{A}_1(0) - g_2 \hat{A}_2(0)]/(2\hat{\lambda}_0^2 \hat{\lambda})$$
$$\hat{\beta}_\pm = [\hat{\lambda}_2^2 g_1 \hat{A}_1(0) + \hat{\lambda}_1^2 g_2 \hat{A}_2(0)]/(2\hat{\lambda}_0^2 \hat{\lambda}) \pm g_1 g_2 \hat{D}(0)/(2\hat{\lambda}\hat{\lambda}_\pm).$$

By using the conservation laws (7) and (13) together with (24) one can obtain

$$\hat{R}_{33}(t) = -\hat{\lambda}_0^2 \hat{P}(t) + \hat{R}_{33}(0)$$
$$\hat{N}_1(t) = \hat{P}_+(t) + \hat{P}_-(t) + \hat{\lambda}_1^2 \hat{P}(t) + \hat{N}_1(0) \qquad (27)$$
$$\hat{N}_2(t) = \hat{P}_+(t) + \hat{P}_-(t) - \hat{\lambda}_2^2 \hat{P}(t) + \hat{N}_2(0).$$

The exact solution (24) of the operator equations (20) and the formulae (27) represent the explicit expressions of time dependence for the level population and photon number operators.

From (25) it is clear that the operators $\hat{\lambda}_+$, $\hat{\lambda}_-$ and $2\hat{\lambda}$ are the quantum electrodynamic expressions for the two-photon Rabi frequencies [11]. Under the condition of one-photon resonance we have $\Delta = 0$, and therefore $\hat{\lambda}_+ = \hat{\lambda}_- = \hat{\lambda}$. In this case there are two branches of the two-photon Rabi frequencies defined by the operators $\hat{\lambda}$ and $2\hat{\lambda}$ [14, 15]. It should be noted that the existence of the 'soft branch' is a characteristic feature of the three-level system. Such a kind of oscillation frequency is absent in the two-level system [17, 19, 20]. Our present results show that the detuning in the case of two-photon resonance leads to the splitting of the 'soft branch' to two branches characterised by the frequency operators $\hat{\lambda}_+ = \hat{\lambda} + \tfrac{1}{2}\Delta$, $\hat{\lambda}_- = \hat{\lambda} - \tfrac{1}{2}\Delta$. This conclusion of the fully quantised theory is in accord with the results of the semiclassical theory [8, 9, 13].

## 4. Time evolution of photon numbers and level populations in the case of quantum initial states

Let $\hat{\rho}(0)$ be a density matrix corresponding to some initial state of the 'atom–field' system. Then the mean values of the level populations and photon numbers are given by

$$\langle \hat{\mathcal{O}}(t) \rangle = \operatorname{Tr} \hat{\mathcal{O}}(t) \hat{\rho}(0) \qquad (28)$$

where $\hat{\mathcal{O}}$ is $\hat{R}_{jj}$ or $\hat{N}_\alpha$.

First of all, let us consider a simple but interesting case when at the initial moment $t = 0$ the atom is on a level $i$ and the field is in a quantum state with definite occupation numbers $|n_1, n_2\rangle$. Then

$$\hat{\rho}(0) = |\{m_0\}\rangle\langle\{m_0\}| \qquad |\{m_0\}\rangle \equiv |i; n_1, n_2\rangle. \qquad (29)$$

One can easily see that the initial state $|\{m_0\}\rangle$ is one of the basis states of the total system. Thus, the density matrix $\hat{\rho}(0)$ has only one non-zero element in the basis representation

$$\rho_{\{m'\},\{m''\}} \equiv \langle\{m'\}|\hat{\rho}(0)|\{m''\}\rangle = \delta_{\{m'\},\{m_0\}}\delta_{\{m''\},\{m_0\}}. \tag{30}$$

On the other hand, the operators $\hat{\lambda}_\alpha$ are diagonal in this representation. So, for an arbitrary operator $\hat{\mathcal{O}}$ and arbitrary function $f(\ )$ we have

$$\begin{aligned}\langle\hat{\mathcal{O}}f(\hat{\lambda}_\alpha)\rangle &= \langle\{m_0\}|\hat{\mathcal{O}}f(\hat{\lambda}_\alpha)|\{m_0\}\rangle \\ &= \langle\{m_0\}|\hat{\mathcal{O}}|\{m_0\}\rangle f(\langle m_0\}|\hat{\lambda}_\alpha|\{m_0\}\rangle) = \langle\hat{\mathcal{O}}\rangle f(\langle\hat{\lambda}_\alpha\rangle).\end{aligned} \tag{31}$$

Below we shall use the following notation $\langle\hat{\mathcal{O}}\rangle \equiv \mathcal{O}$. Now by using the relation (31) we can obtain from (24) and (27) that

$$\begin{aligned}R_{11}(t) &= -2\mu_+\sin^2\tfrac{1}{2}\lambda_+t - 2\mu_-\sin^2\tfrac{1}{2}\lambda_-t - 2\lambda_1^2\mu\sin^2\lambda t + R_{11}(0)\\ R_{22}(t) &= 2\mu_+\sin^2\tfrac{1}{2}\lambda_+t + 2\mu_-\sin^2\tfrac{1}{2}\lambda_-t - 2\lambda_2^2\mu\sin^2\lambda t + R_{22}(0)\\ R_{33}(t) &= 2\lambda_0^2\mu\sin^2\lambda t + R_{33}(0)\\ N_1(t) &= -2\mu_+\sin^2\tfrac{1}{2}\lambda_+t - 2\mu_-\sin^2\tfrac{1}{2}\lambda_-t - 2\lambda_1^2\mu\sin^2\lambda t + n_1\\ N_2(t) &= -2\mu_+\sin^2\tfrac{1}{2}\lambda_+t - 2\mu_-\sin^2\tfrac{1}{2}\lambda_-t + 2\lambda_2^2\mu\sin^2\lambda t + n_2.\end{aligned} \tag{32}$$

Here the frequencies $\lambda_+$, $\lambda_-$ and $2\lambda$ of the two-photon Rabi oscillations in the system are defined by

$$\lambda = (\lambda_0^2 + \tfrac{1}{4}\Delta^2)^{1/2} \qquad \lambda_+ = \lambda + \tfrac{1}{2}\Delta \qquad \lambda_- = \lambda - \tfrac{1}{2}\Delta \tag{33}$$

where

$$\lambda_0 = (\lambda_1^2 + \lambda_2^2)^{1/2} \qquad \lambda_1 = g_1(n_1 - R_{11}(0) + 1)^{1/2} \qquad \lambda_2 = g_2(n_2 + R_{22}(0))^{1/2}. \tag{34}$$

The amplitudes of the oscillations are found from (26) to be

$$\begin{aligned}\mu &= [\lambda_1^2R_{11}(0) + \lambda_2^2R_{22}(0) - \lambda_0^2R_{33}(0)]/(2\lambda_0^2\lambda^2)\\ \mu_\pm &= [R_{11}(0) - R_{22}(0)]\lambda_1^2\lambda_2^2/(\lambda_0^2\lambda\lambda_\pm)\\ \beta &= \beta_+ = \beta_- = 0.\end{aligned} \tag{35}$$

For the sake of eliminating the above-mentioned fast oscillations and obtaining the time-average values of the mean level populations and photon numbers, we use the following procedure:

$$\bar{\mathcal{O}} = \frac{1}{2T}\int_{t-T}^{t+T}\mathcal{O}(t')\,dt' \qquad T \gg \lambda^{-1} \tag{36}$$

for $\mathcal{O}(t) = R_{jj}(t),\ N_\alpha(t)$.

Then, in compliance with (32) we have

$$\begin{aligned}\bar{R}_{11} &= -(\mu_+ + \mu_- + \lambda_1^2\mu) + R_{11}(0)\\ \bar{R}_{22} &= \mu_+ + \mu_- - \lambda_2^2\mu + R_{22}(0)\\ \bar{R}_{33} &= \lambda_0^2\mu + R_{33}(0)\\ \bar{N}_1 &= -(\mu_+ + \mu_- + \lambda_1^2\mu) + n_1\\ \bar{N}_2 &= -(\mu_+ + \mu_- - \lambda_2^2\mu) + n_2.\end{aligned} \tag{37}$$

Let us now concretise the initial condition (29) and find the values of the frequencies and amplitudes corresponding to the cases when the atom is initially in the state 1, 2, 3, respectively.

*Case 1.* Let the atom be in the unexcited state $|1\rangle$ at $t=0$, i.e. $|\{m_0\}\rangle = |1; n_1, n_2\rangle$. In this case we have $R_{11}(0)=1$, $R_{22}(0)=R_{33}(0)=0$. From (33)-(35) it follows that

$$\begin{aligned}
&\lambda = W(\{n_\alpha\}) \qquad \lambda_{0,\pm} = W_{0,\pm}(\{n_\alpha\}) \\
&\lambda_1 = g_1\sqrt{n_1} \qquad \lambda_2 = g_2\sqrt{n_2} \\
&\mu = \frac{g_1^2 n_1}{2W_0^2(\{n_\alpha\})\,W^2(\{n_\alpha\})} \\
&\mu_\pm = \frac{g_1^2 g_2^2 n_1 n_2}{W_0^2(\{n_\alpha\})\,W(\{n_\alpha\})\,W_\pm(\{n_\alpha\})}
\end{aligned} \tag{38}$$

where we have introduced the notation

$$\begin{aligned}
&W_0(\{n_\alpha\}) \equiv W_0(n_1, n_2) = (g_1^2 n_1 + g_2^2 n_2)^{1/2} \\
&W(\{n_\alpha\}) \equiv W(n_1, n_2) = (g_1^2 n_1 + g_2^2 n_2 + \tfrac{1}{4}\Delta^2)^{1/2} \\
&W_\pm(\{n_\alpha\}) \equiv W_\pm(n_1, n_2) = (g_1^2 n_1 + g_2^2 n_2 + \tfrac{1}{4}\Delta^2)^{1/2} \pm \tfrac{1}{2}\Delta.
\end{aligned} \tag{39}$$

*Case 2.* Let the atom be in the upper state 2 at $t=0$, thus $|\{m_0\}\rangle = |2; n_1, n_2\rangle$. Then we have $R_{22}(0)=1$, $R_{11}(0)=R_{33}(0)=0$. Equations (33)-(35) in this case give

$$\begin{aligned}
&\lambda = W(\{n_\alpha+1\}) \qquad \lambda_{0,\pm} = W_{0,\pm}(\{n_\alpha+1\}) \\
&\lambda_1 = g_1(n_1+1)^{1/2} \qquad \lambda_2 = g_2(n_2+1)^{1/2} \\
&\mu = \frac{g_2^2(n_2+1)}{2W_0^2(\{n_\alpha+1\})\,W^2(\{n_\alpha+1\})} \\
&\mu_\pm = -\frac{g_1^2 g_2^2 (n_1+1)(n_2+1)}{W_0^2(\{n_\alpha+1\})\,W(\{n_\alpha+1\})\,W_\pm(\{n_\alpha+1\})}.
\end{aligned} \tag{40}$$

*Case 3.* Let the atom be in the immediate state 3 at $t=0$, i.e. $|\{m_0\}\rangle = |3; n_1, n_2\rangle$. In this case we have $R_{33}(0)=1$, $R_{11}(0)=R_{22}(0)=0$. From (33)-(35) one can find that

$$\begin{aligned}
&\lambda = W(n_1+1, n_2) \qquad \lambda_{0,\pm} = W_{0,\pm}(n_1+1, n_2) \\
&\lambda_1 = g_1(n_1+1)^{1/2} \qquad \lambda_2 = g_2 n_2^{1/2} \\
&\mu = -\frac{1}{2W^2(n_1+1, n_2)} \qquad \mu_\pm = 0.
\end{aligned} \tag{41}$$

Note that the expressions (32) together with (38), (40) and (41) are in compliance with the results of [9] and [11].

To determine the transition probabilities of the atom, let us introduce the Schrödinger representation with a wavefunction of the total system $|\psi(t)\rangle$, where $|\psi(0)\rangle = |i; n_1, n_2\rangle$. Then, the probability of finding the atom on its $j$th level at time $t$ as a result of the transition $i \to j$ initiated by the $n_1 \oplus n_2$ photon field can be defined by the formula

$$P(t; i\to j) = \sum_{n_1', n_2'} |\langle \psi(t)|j; n_1', n_2'\rangle|^2 \tag{42}$$

where

$$|\psi(0)\rangle = |i; n_1, n_2\rangle. \tag{43}$$

It is seen that under the initial condition (43) the population $R_{jj}(t)$ of level $j$ is equal to the probability $P(t; i\to j)$. Hence, by using (32) together with (38), (40) and (41) one can determine the probabilities of various transitions in the system. In particular, for the two-photon processes of absorption $(1\to 2)$ and emission $(2\to 1)$ one obtains [21]

$$P(t;1\to 2)=\frac{2g_1^2g_2^2n_1n_2}{W_0^2(\{n_\alpha\})\,W(\{n_\alpha\})}\left(\frac{1}{W_+(\{n_\alpha\})}\sin^2\frac{W_+(\{n_\alpha\})t}{2}+\frac{1}{W_-(\{n_\alpha\})}\sin^2\frac{W_-(\{n_\alpha\})t}{2}-\frac{1}{2W(\{n_\alpha\})}\sin^2[W(\{n_\alpha\})t]\right)$$

$$P(t;2\to 1)=\frac{2g_1^2g_2^2(n_1+1)(n_2+1)}{W_0^2(\{n_\alpha+1\})\,W(\{n_\alpha+1\})}\times\left(\frac{1}{W_+(\{n_\alpha+1\})}\sin^2\frac{W_+(\{n_\alpha+1\})t}{2}+\frac{1}{W_-(\{n_\alpha+1\})}\sin^2\frac{W_-(\{n_\alpha+1\})t}{2}-\frac{1}{2W(\{n_\alpha+1\})}\sin^2[W(\{n_\alpha+1\})t]\right) \tag{44}$$

For the one-photon transitions $3\rightleftarrows\alpha$ $(\alpha=1,2)$ we find

$$\begin{aligned}
P(t;1\to 3)&=\frac{g_1^2n_1}{W^2(\{n_\alpha\})}\sin^2[W(\{n_\alpha\})t]\\
P(t;2\to 3)&=\frac{g_2^2(n_2+1)}{W^2(\{n_\alpha+1\})}\sin^2[W(\{n_\alpha+1\})t]\\
P(t;3\to 1)&=\frac{g_1^2(n_1+1)}{W^2(n_1+1,n_2)}\sin^2[W(n_1+1,n_2)t]\\
P(t;3\to 2)&=\frac{g_2^2n_2}{W^2(n_1+1,n_2)}\sin^2[W(n_1+1,n_2)t]
\end{aligned} \tag{45}$$

The expressions (44) and (45) are in compliance with the results of [9] and [11].

## 5. Time evolution of photon numbers and level populations in the case of arbitrary initial field

Now we consider the case when the field is initially in some state described by the density matrix $\hat{\rho}_F$ whereas the atom is in level $i$. The total density matrix of the 'atom-field' system is

$$\hat{\rho}(0)=|i\rangle\langle i|\otimes\hat{\rho}_F. \tag{46}$$

In the case of initially coherent field the matrix $\hat{\rho}_F$ takes the form [22]

$$\hat{\rho}_F=|Z_1,Z_2\rangle\langle Z_1,Z_2| \tag{47}$$

where the coherent state $|Z_1,Z_2\rangle$ is defined by

$$|Z_1,Z_2\rangle\equiv\sum_{n_1,n_2}\exp\left(-\frac{|Z_1|^2+|Z_2|^2}{2}\right)\frac{Z_1^{n_1}Z_2^{n_2}}{(n_1!\,n_2!)^{1/2}}|n_1,n_2\rangle. \tag{48}$$

In the case of an initially chaotic field $\hat{\rho}_F$ is

$$\hat{\rho}_F = \exp(-\beta \hat{H}_F) / \operatorname*{Tr}_{(F)} \exp(-\beta \hat{H}_F). \tag{49}$$

Here the field Hamiltonian $\hat{H}_F$ is given by (3) and $\beta^{-1}$ is the temperature of the initial field.

It is seen from (24)-(27) that the operators $\hat{R}_{jj}(t)$ and $\hat{N}_\alpha(t)$ are diagonal in the state subspaces $\{|1; n_1, n_2\rangle\}$, $\{|2; n_1, n_2\rangle\}$ and $\{|3; n_1, n_2\rangle\}$, i.e. for $\hat{\mathcal{O}} = \hat{R}_{jj}(t)$, $\hat{N}_\alpha(t)$ one has

$$\langle i; n_1', n_2' | \hat{\mathcal{O}} | i; n_1'', n_2'' \rangle = \delta_{n_1', n_1''} \delta_{n_2', n_2''} \langle i; n_1', n_2' | \hat{\mathcal{O}} | i; n_1', n_2' \rangle. \tag{50}$$

Hence the mean value of $\hat{\mathcal{O}}$ in the initial state (46) is found to be

$$\langle \hat{\mathcal{O}} \rangle = \operatorname{Tr} \hat{\mathcal{O}} \hat{\rho}(0) = \sum_{n_1, n_2} \langle \hat{\mathcal{O}} \rangle_{in_1 n_2} P(n_1, n_2) \tag{51}$$

where $\langle \hat{\mathcal{O}} \rangle_{in_1 n_2}$ is the mean value of $\hat{\mathcal{O}}$ in the initial state (29)

$$\langle \hat{\mathcal{O}} \rangle_{in_1 n_2} = \langle i; n_1, n_2 | \hat{\mathcal{O}} | i, n_1, n_2 \rangle \tag{52}$$

and $P(n_1, n_2)$ is the weight factor defined by the field density matrix $\hat{\rho}_F$

$$P(n_1, n_2) \equiv \langle n_1, n_2 | \hat{\rho}_F | n_1, n_2 \rangle. \tag{53}$$

Thus, by using relation (51) and (32), together with (38), (40) and (41), one can obtain the mean values of the level populations and photon numbers in the general case. In particular, we find

$$\begin{aligned}
R_{11}(t) &= N_1(t) - N_1(0) + 1 \\
&= 1 - \sum_{n_1, n_2 = 0}^{\infty} \frac{2 g_1^2 g_2^2 n_1 n_2}{W_0^2(\{n_\alpha\}) W(\{n_\alpha\})} \left( \frac{1}{W_+(\{n_\alpha\})} \sin^2 \frac{W_+(\{n_\alpha\}) t}{2} \right. \\
&\quad \left. + \frac{1}{W_-(\{n_\alpha\})} \sin^2 \frac{W_-(\{n_\alpha\}) t}{2} \right) P(n_1, n_2) \\
&\quad - \sum_{n_1, n_2 = 0}^{\infty} \frac{g_1^4 n_1^2}{W_0^2(\{n_\alpha\}) W^2(\{n_\alpha\})} \sin^2 [W(\{n_\alpha\}) t] P(n_1, n_2) \\
R_{22}(t) &= N_2(0) - N_2(t) \\
&= \sum_{n_1, n_2 = 0}^{\infty} \frac{2 g_1^2 g_2^2 n_1 n_2}{W_0^2(\{n_\alpha\}) W(\{n_\alpha\})} \left( \frac{1}{W_+(\{n_\alpha\})} \sin^2 \frac{W_+(\{n_\alpha\}) t}{2} \right. \\
&\quad \left. + \frac{1}{W_-(\{n_\alpha\})} \sin^2 \frac{W_-(\{n_\alpha\}) t}{2} - \frac{1}{2 W(\{n_\alpha\})} \sin^2 [W(\{n_\alpha\}) t] \right) P(n_1, n_2) \\
R_{33}(t) &= \sum_{n_1, n_2 = 0}^{\infty} \frac{g_1^2 n_1}{W^2(\{n_\alpha\})} \sin^2 [W\{n_\alpha\}) t] P(n_1, n_2)
\end{aligned} \tag{54}$$

for the case $\hat{\rho}(0) = |1\rangle\langle 1| \otimes \hat{\rho}_F$, when the atom is initially on the lower level 1. For the other case when the atom is initially on the upper level 2, and therefore $\hat{\rho}(0) = |2\rangle\langle 2| \otimes \hat{\rho}_F$, we obtain

$$
\begin{aligned}
R_{11}(t) &= N_1(t) - N_1(0) \\
&= \sum_{n_1,n_2=0}^{\infty} \frac{2g_1^2 g_2^2 (n_1+1)(n_2+1)}{W_0^2(\{n_\alpha+1\})\, W(\{n_\alpha+1\})} \left( \frac{1}{W_+(\{n_\alpha+1\})} \sin^2 \frac{W_+(\{n_\alpha+1\})t}{2} \right. \\
&\quad + \frac{1}{W_-(\{n_\alpha+1\})} \sin^2 \frac{W_-(\{n_\alpha+1\})t}{2} \\
&\quad \left. - \frac{1}{2W(\{n_\alpha+1\})} \sin^2[W(\{n_\alpha+1\})t] \right) P(n_1, n_2) \\
R_{22}(t) &= 1 + N_2(0) - N_2(t) \\
&= 1 - \sum_{n_1,n_2=0}^{\infty} \frac{2g_1^2 g_2^2 (n_1+1)(n_2+1)}{W_0^2(\{n_\alpha+1\})\, W(\{n_\alpha+1\})} \left( \frac{1}{W_+(\{n_\alpha+1\})} \sin^2 \frac{W_+(\{n_\alpha+1\})t}{2} \right. \\
&\quad \left. + \frac{1}{W_-(\{n_\alpha+1\})} \sin^2 \frac{W_-(\{n_\alpha+1\})t}{2} \right\} P(n_1, n_2) \\
&\quad - \sum_{n_1,n_2=0}^{\infty} \frac{g_2^4 (n_2+1)^2}{W_0^2(\{n_\alpha+1\})\, W^2(\{n_\alpha+1\})} \sin^2[W(\{n_\alpha+1\}t] P(n_1, n_2) \\
R_{33}(t) &= \sum_{n_1,n_2=0}^{\infty} \frac{g_2^2 (n_2+1)}{W^2(\{n_\alpha+1\})} \sin^2[W(\{n_\alpha+1\})t] P(n_1, n_2).
\end{aligned}
\tag{55}
$$

For illustration we calculate the time variation of the photon numbers $\delta N_\alpha(t) = N_\alpha(t) - N_\alpha(0)$ for the case when the atom is initially unexcited on level 1 and the field is in state (47) or (49).

### 5.1. *Initially coherent field*

First we consider the case when the field is initially in the coherent state (47). In this case, according to (48) and (53), we have

$$
P(n_1, n_2) = \exp[-(\bar{n}_1 + \bar{n}_2)] \bar{n}_1^{n_1} \bar{n}_2^{n_2} / (n_1!\, n_2!). \tag{56}
$$

Here $\bar{n}_1 = |Z_1|^2$ and $\bar{n}_2 = |Z_2|^2$ are the mean photon numbers in modes 1 and 2, respectively, at the initial time of the interaction.

Substituting (56) into (54) and by using (39) we can now calculate the time evolution of $\delta N_1(t)$ and $\delta N_2(t)$. The results of calculations for the case $g_1 = g_2 = g$, $\Delta = 0$, $\bar{n}_1 = \bar{n}_2 = 5$ are shown in figure 2 by sketching the simple lines coupling the peak points obtained. From the figure we see that the expectation values of the photon numbers and, hence, also the expectation values of the level populations have oscillations which decay rapidly at short times but periodically regenerate to larger amplitudes on a much longer timescale. Such quantum collapses and revivals in a loss-free system have been predicted in the coherent state Jaynes–Cummings model [23] and in the lambda configuration three-level model [14, 24].

Note that the maximum of the second revival of $\delta N_\alpha(t)$ in the case considered in figure 2 is remarkably larger than the maximum of the first revival. Such a feature is absent in the Jaynes–Cummings model [23]. We can explain this feature of the

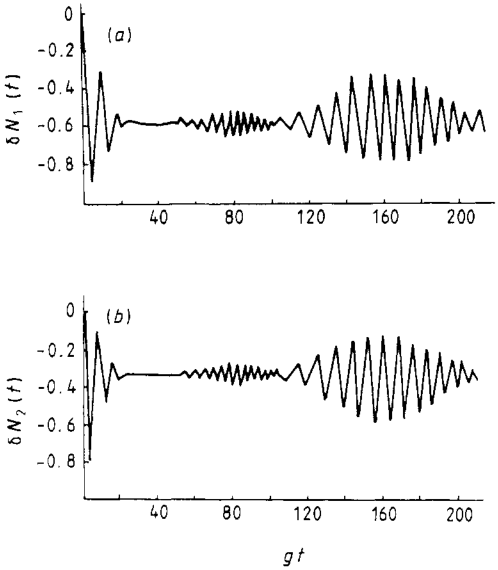


**Figure 2.** Time evolution of the photon numbers in the case of the initially coherent field (for $g_1 = g_2 = g$, $\Delta = 0$, $\bar{n}_1 = \bar{n}_2 = 5$).

three-level two-mode results by the existence of different kinds of revival which have different periods and maxima owing to the variety of the Rabi frequency branches $W_\pm(n_1, n_2)$, i.e. $\hat{\lambda}_\pm$ (or $\hat{\lambda}$), and $2W(n_1, n_2)$, i.e. $2\hat{\lambda}$, of the sums in (54). The second revival of $\delta N_\alpha(t)$ in figure 2 belongs to the kind defined by the sums with the oscillation factors $\sin^2(\frac{1}{2}W_\pm t)$ whilst the previous small revival pertains to the kind defined by the sums with the oscillation factors $\sin^2(Wt)$. Hence we can easily understand why the monotonous decreasing behaviour of the revival maxima in the model with one Rabi frequency branch [23] does not occur in the model considered here.

### 5.2. *Initially chaotic field*

We consider now the case when the field is initially in the chaotic (thermal) state (49). Then, the weight factor $P(n_1, n_2)$ takes the form

$$P(n_1, n_2) = Z^{-1} \exp[-\beta(\hbar\omega_1 n_1 + \hbar\omega_2 n_2)] \tag{57}$$

where

$$Z^{-1} = [1 - \exp(-\beta\hbar\omega_1)][1 - \exp(-\beta\hbar\omega_2)]. \tag{58}$$

The time behaviour of $\delta N_1(t)$ and $\delta N_2(t)$ has been calculated for $g_1 = g_2 = g$, $\hbar\omega_1\beta = \hbar\omega_2\beta = 0.2$, $\Delta = 0$, and is sketched by the simple lines connecting the peak points in figure 3. The collapse at short times, the flat feature before the revival of the rapid oscillations and the long and non-regular (chaotic) characters of the last are seen. Such a behaviour has been noted in [25] for the Jaynes–Cummings model in the case of the initially chaotic field.

Thus, we see from the two examples described above that the quantum collapse and revival are possible in the loss-free three-level two-mode model. These effects are due to the spread of Rabi frequencies and the quantum nature of the cavity radiation

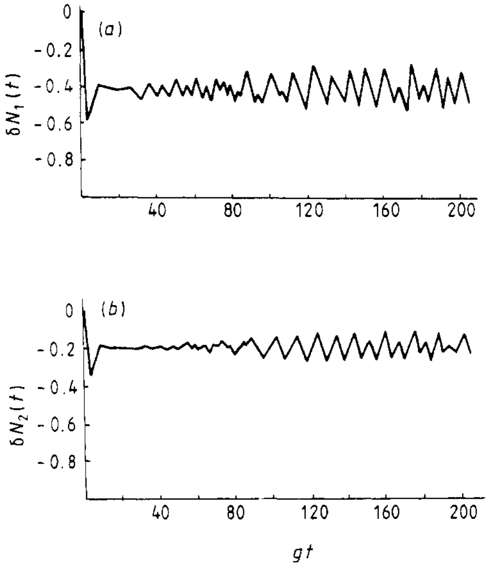


**Figure 3.** Time evolution of the photon numbers in the case of the initially chaotic field (for $g_1 = g_2 = g$, $\Delta = 0$, $\hbar\omega_1\beta = \hbar\omega_2\beta = 0.2$).

field which manifests itself in the discreteness of the photon numbers in statistical averaging [25]. Therefore, the influences of the level and mode numbers appear only in the quantitative characteristics and some fine features of the phenomena. It is why our results look very similar to those of the Jaynes–Cummings model [23, 25].

## 6. Conclusion

In this paper the operator equations for the ladder configuration three-level atom interacting with a two-mode quantised radiation field have been solved explicitly in the two-photon resonance condition. The quantum electrodynamic expressions of two-photon Rabi frequencies have been found. The time evolution of the photon numbers and level populations has been examined. It has been shown that the quantum revival and collapse are possible in the loss-free three-level two-mode model. The large size of the maximum of the second revival compared with that of the first has been noted in the case of the initially coherent field. We emphasise that some of our particular results can easily be obtained by diagonalising the Hamiltonian and using the dressed-state formalism. More detailed study of the revivals and collapses and an investigation of the photon statistical properties will be the subject of a subsequent paper.

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